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SIMULTANEOUS ESTIMATION OF TWO BOUNDARY CONDITIONS IN A TWO-DIMENSIONAL HEAT CONDUCTION PROBLEM

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Abstract - A simultaneous estimation of two boundary conditions in a heat conduction problem is proposed by a numerical approach. The aim is to estimate the evolution of the distributions of the unknown surface heat fluxes from the transient temperature histories taken with several sensors inside a two-dimensional specimen. The temperatures are known on two lines in the finite body. The numerical algorithm of this inverse heat conduction problem is based on the iterative regularization method and on the conjugate gradient method. The utilization of two descent parameters is at the origin of this method. For each boundary heat flux, a descent parameter is computed and an optimal choice of the matrix of the descent parameters is used, showing an increase in the convergence rate. All numerical simulations are performed for the two-dimensional linear heat conduction problem. The effects of sensor positions and the magnitude of measurement errors on the inverse solutions are discussed. Numerical results for some representative cases suggest that heat fluxes and temperature can be predicted well by this method.

NOMENCLATURE

a,b	dimensions of the specimen
bx	random noise
f_n	measured temperature at position (X_n, Y_n)
h_i	heat transfer coefficient
N	number of temperature sensors
N_t	number of time steps
N_x, N_y	number of nodes in the *x, y* direction
T, T_a, T_0	temperature, ambient temperature, initial temperature
$q_i(y,t)$	heat flux density
t, t_f	time, final time
α, λ	thermal diffusivity, thermal conductivity
δ^2	estimated error or criterion
$\Delta x, \Delta y, \Delta t$	spatial grid in the *x, y* directions, time step
ΔFo	delta Fourier number
ψ	adjoint variable.

1. INTRODUCTION

Most heat transfer processes occurring in industrial applications require accurate knowledge of the thermal properties of the material and surface conditions. Practically, measurements are often made of temperature and displacement, etc. Thereafter, physical quantities and surface conditions may to be estimated from these measurements. Such problems are called inverse problems and recently have become an interesting subject. To date, various methods have been developed for the analysis of the inverse heat conduction problems involving the estimation of the surface conditions from measured temperatures inside the material. Most of the analytical and numerical methods only deal with one- two- or three-dimensional inverse heat conduction problems (IHCP) to estimate a single surface condition [1-7]. However, a few works estimate more than one surface condition in two and three dimensional problems.

Chen *et al.* [8] have applied the Laplace transform technique and finite-difference method with a sequential in time concept. The least square scheme is proposed to predict the unknown surface temperature of two sided boundary conditions for a two dimensional inverse heat conduction problem.

Hsu *et al.* [9] have used the finite difference method in conjunction with the linear least squares method to estimate the one-sided and two-sided boundary conditions in two-dimensional IHCP. In their work, they suppose that the functional form of the estimated surface temperature is given *a priori* and then parametrized. However, the effect of the measurement errors on the surface temperature cannot be neglected. Recently, Loulou *et al.* [10] used the iterative regularization method in one dimensional IHCP problem to estimate a combination of two kinds of surface boundary conditions.

In this work, we propose a simultaneous estimation of transient distributions of two boundary heat conditions by using the iterative regularization method and transient temperature histories taken with several sensors inside a two-dimensional specimen.

2. PHYSICAL PROBLEM

The specimen is a rectangular plate heated by two unknown heat fluxes at the active opposite surfaces. The others sides are insulated or submitted to convection heat transfer with the ambient.

The following hypotheses have been taken into account:

- Thermo-physical properties are assumed to be constant
- Heat transfer is two-dimensional
- Heat fluxes are variable with space and time.

Under these conditions, the heat transfer process in the specimen can be described by the following system of equations:

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), 0 < x < a \,,\; 0 < y < b \,,\; 0 < t \le t_f \tag{1}$$

$$-\lambda \frac{\partial T(0,y,t)}{\partial x} = q_1(y,t) \tag{2}$$

$$-\lambda \frac{\partial T(a,y,t)}{\partial x} = q_2(y,t) \tag{3}$$

$$-\lambda \frac{\partial T(x,0,t)}{\partial y} = h_1(T_a - T(x,0,t)) \tag{4}$$

$$-\lambda \frac{\partial T(x,b,t)}{\partial y} = h_2(T(x,b,t) - T_a) \tag{5}$$

$$T(x,y,0) = T_0 \tag{6}$$

In the model (1)-(6), the heat fluxes density $q_1(y,t)$ and $q_2(y,t)$ are unknowns. To obtain additional information about the temperature distribution in the specimen, temperature histories are measured in the specimen at a certain number of points N with coordinates $(x,y) = (X_n, Y_n)$, $n = 1,2,...,N$, distributed on two lines parallel to the active surfaces, namely

$$T_{meas}(X_n, Y_n, t) = f_n(t)\,,\; n = 1,2,...,N \tag{7}$$

where: $X_n = x_\alpha$ for $1 \le n \le N/2$

$X_n = x_\beta$ for $N/2 < n \le N$

This information, together with the model (1)-(6), is used to solve the inverse problem.

3. INVERSE PROBLEM

To build a computational algorithm, we use the variational formulation of the inverse problem of interest. The problem is to find unknown functions $q_1(y,t)$ and $q_2(y,t)$ for which temperature histories computed from the mathematical model (1) to (6) at the sensor locations would be close to measured histories. This leads to the problem of minimizing the residual functional:

$$J(q_1,q_2) = \sum_{n=1}^{N} \int_0^{t_f} \left[T(X_n, Y_n, t; q_1, q_2) - f_n(t)\right]^2 dt \tag{8}$$

where $T(X_n, Y_n, t; q_1, q_2)$, $n = 1,2,...,N$, are temperature histories computed at the sensor locations with given heat fluxes $q_1(y,t)$ and $q_2(y,t)$.

The conjugate gradient method is used to solve this inverse heat conduction problem:

$$q_i^{(k+1)}(y,t) = q_i^{(k)}(y,t) + \gamma_i^{(k)} D_i^{(k)}(y,t),\ i=1,2 \tag{9}$$

where: $D_i^{(k)}(y,t), i=1,2$, is the descent direction defined by the relation:

$$D_i^{(k)}(y,t) = -J'^{(k)}_{q_i}(y,t) + \beta_i^{(k)} D^{(k-1)}(y,t) \tag{10}$$

$$J'^{(k)}_{q_i}(y,t) = \psi^{(k)}(x_i,y,t) \ \ i=1,2 \tag{11}$$

where $x_i = 0$ for the heat flux $q_1(y,t)$ and $x_i = a$ for the heat flux $q_2(y,t)$.

In the expression (10), $J'^{(k)}_{q_i}(y,t)$ represent the gradient of the residual functional (8) and $\psi(x,y,t)$ represents the solution of the adjoint problem [3]. $\beta_i^{(k)}$, $i=1,2$, is the conjugate parameter defined by:

$$\beta_i^{(k)} = \frac{\left\langle J'^{(k)}_{q_i} - J'^{(k-1)}_{q_i}, J'^{(k)}_{q_i} \right\rangle}{\left\| J'^{(k)}_{q_i} \right\|} \ \ \beta_i^{(0)} = 0 \tag{12}$$

where $\langle , \rangle$ is the scalar product and $\| \ \|$ the norm defined in the working space.

$\gamma_i^{(k)}$, $i=1,2$, is the descent parameter computed from the temperature variation for each heat flux and estimated by a linear approximation of the residual functional. The optimal value of the vector $\gamma^{(k)} = \left[\gamma_1^{(k)}, \gamma_2^{(k)}\right]^T$ is obtained by solving the minimization problem:

$$\gamma^{(k)} = \left[\gamma_1^{(k)}, \gamma_2^{(k)}\right]^T : J(q^{(k)} + \gamma^{(k)} D^{(k)}) \to \min_{\gamma > 0}$$

and by solving a matrix system of two dimensions.

To obtain a stable solution of the inverse problem, the iterative regularization method is used [3]. The main idea is to terminate the iterative procedure when the residual criterion is satisfied:

$$J(q_1^{(k)}, q_2^{(k)}) \approx \delta^2 \tag{13}$$

where δ^2 is the total (integrated) measurement error defined by:

$$\delta^2 = \sum_{n=1}^{N} \int_0^{t_f} \sigma_n^2(t) dt \tag{14}$$

and $\sigma_n^2(t)$ is an estimate of the time-dependent standard deviation for the nth measured temperature history. This procedure gives the most stable solution. The number k^* of the last iteration is the regularization parameter of the method.

One iteration of the numerical algorithm includes the following steps:

- Solution of the direct problem and computation of the residual functional,
- Verification of the residual criterion,
- Solution of the adjoint problem and computation of the residual functional gradient for each unknown heat flux,
- Computation of the descent direction for each heat flux boundary condition,
- Solution of the problem for temperature variations and computation of the optimal descent parameter for each boundary condition,
- Calculation of the two estimated heat fluxes.

4. RESULTS AND DISCUSSION

To simulate the numerical solution, we supposed in the problem (1) to (6) that: $\lambda = 1$ W/(m.C), $\alpha = 1$ m^2./s, $a = b = 1$ m, $T_0 = T_a = 0$ °C, $h_1 = h_2 = 10$ W/(m^2.C).

The measured temperature histories were simulated numerically and uniformly distributed inside the specimen on two lines (L_α) and (L_β) parallel to the active surfaces and defined, respectively, by the coordinates (x_α, y_j) and (x_β, y_j), $y_j = j\Delta y$, $j = 1,\dots,N/2$, $x_\alpha = \alpha\Delta x$, $x_\beta = \beta\Delta x$, $\Delta x = \dfrac{a}{N_x}$, $\Delta y = \dfrac{b}{N_y}$, $N_x = N_y = 10$.

A random noise of $bx\,\%$ in the maximal temperature value is applied to the simulated temperatures.

The alternative direction implicit method (ADI) is used to solve different boundary problems.

In the first test case, we have assumed:

$$q_1(y,t) = q_2(y,t) = q_0 f(y) g(t)$$

where: $f(y) = \frac{y}{b}(1-\frac{y}{b})$ and $g(t) = \frac{t}{t_f}(1-\frac{t}{t_f})$, $0 \le y \le b$, $0 \le t \le t_f$, $q_0 = 4000\ W/m^2$, $\Delta t = 0{,}01\ s$, $t_f = 1\ s$.

The exact and estimated heat fluxes and the simulated and estimated temperatures $T(x,y,t_f)$, $T(x,b/2,t)$ and $T(a/2,y,t)$ are presented in Figures 1a-d and 2a-d, respectively. The estimated results are obtained for the following conditions:
Number of the sensors: $N = 22$,
Position of the first line (L_α): $x_\alpha = 2\Delta x$,
Position of the second line (L_β): $x_\beta = 9\Delta x$,
Random noise: $bx = 1\%$.

A perfect agreement exists at the final time between exact and estimated heat flux and temperatures. A small difference is shown, during the intermediate transient regime, in the axial direction y close the boundaries $y = 0$ and $y = b$.

In the following steps, we have chosen to present only the influences of the sensor locations for symmetrical and non symmetrical positions of lines (L_α) and (L_β), as well as the influence of the noise of measurement on the estimated results. Then, we finish this study by presenting an example where the two boundary heat fluxes are different.

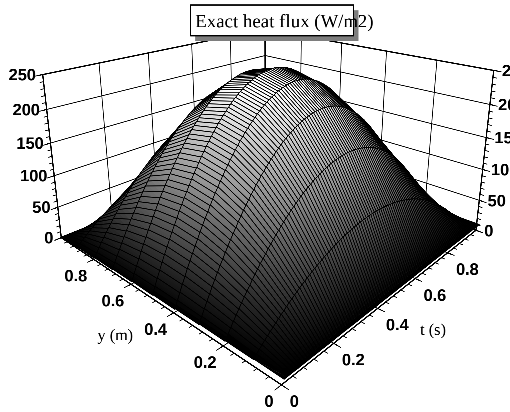

Figure 1a.
Exact heat flux, $q_1(y,t) = q_2(y,t)$.

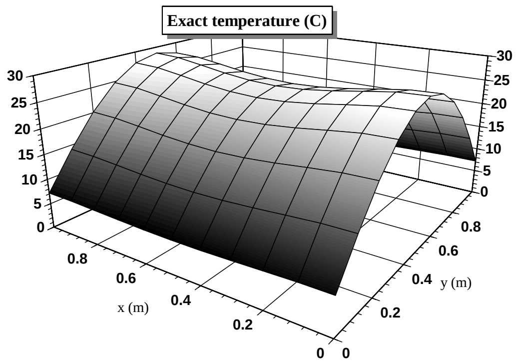

Figure 1b.
Exact temperature $T(x,y,t_f)$ at final time.

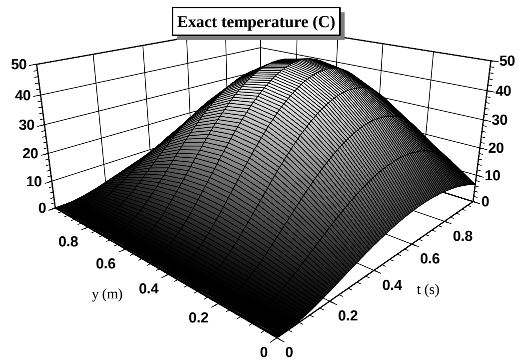

Figure 1c.
Exact temperature $T(x_c,y,t)$ for $x_c = a/2$.

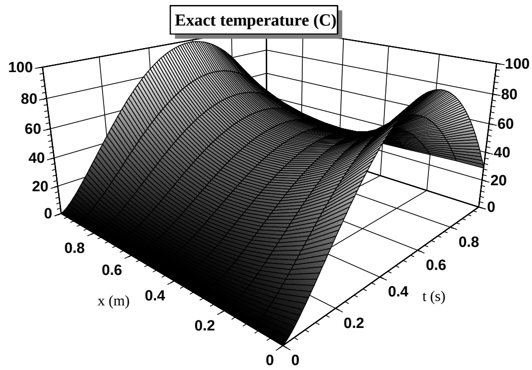

Figure 1d.
Exact temperature $T(x,y_c,t)$ for $y_c = b/2$.

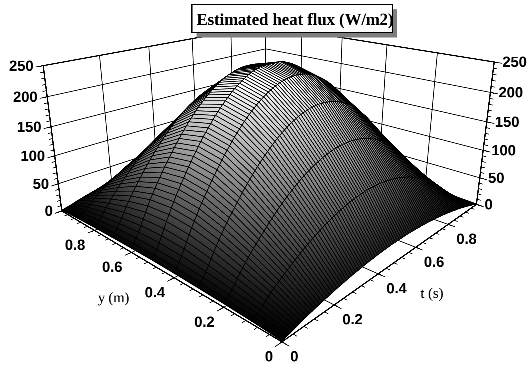

Figure 2a.1.
Estimated heat flux $q_1(y,t)$.

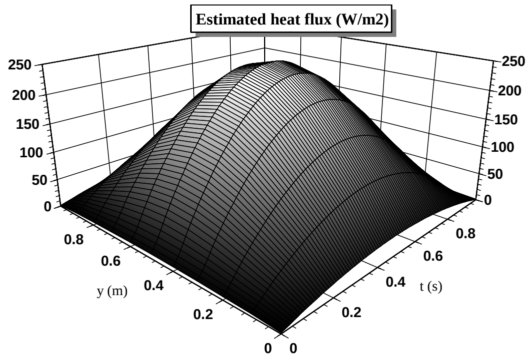

Figure 2a.2.
Estimated heat flux $q_2(y,t)$.

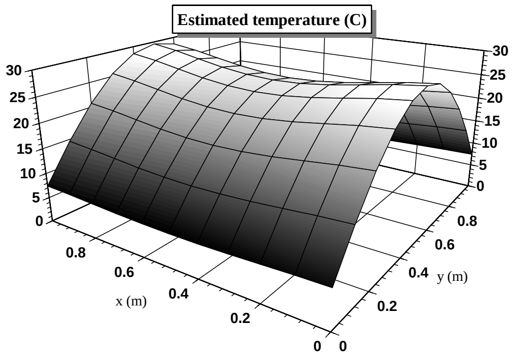

Figure 2b.
Estimated temperature $T_{es}(x,y,t_f)$ at final time.

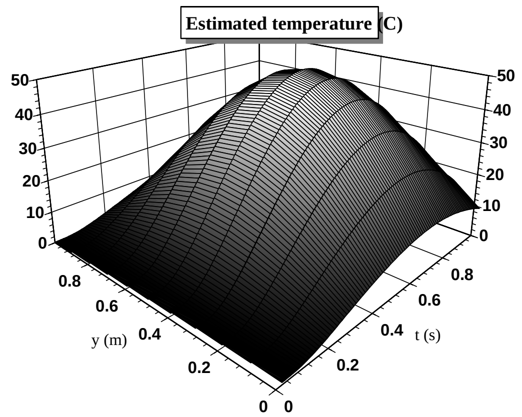

Figure 2c.
Estimated temperature $T_{es}(x_c,y,t)$ for $x_c = a/2$.

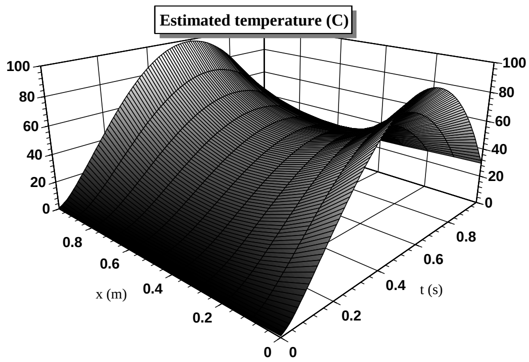

Figure 2d.
Estimated temperature $T_{es}(x,y_c,t)$ for $y_c = b/2$.

4.1. INFLUENCE OF SENSOR LOCATIONS

For $bx = 1\%$ and $N = 22$, the estimated results are presented in Figures 3a-d for five symmetrical positions (x_α, x_β) of the two sensor lines. The corresponding delta Fourier numbers ($\Delta Fo_1 = \dfrac{\alpha \Delta t}{x_\alpha^2}$, $\Delta Fo_2 = \dfrac{\alpha \Delta t}{(a - x_\beta)^2}$) and the residual criterion δ^2 are presented in Table 1.

Table 1.

Position	(x_α, x_β)	$\Delta Fo_1 = \Delta Fo_2$	δ^2
1	$(0,1)$	∞	2.69478
2	$(0.1,0.9)$	1	1.70055
3	$(0.2,0.8)$	0.25	1.14183
4	$(0.3,0.7)$	0.1111	0.83695
5	$(0.4,0.6)$	0.0625	0.68446

The results show a perfect agreement between the evolution of exact and estimated heat fluxes for the first position $(0,1)$ and the second position $(0.1,0.9)$. Good agreement is noted also between the distributions in the estimated heat fluxes $q_1(y,t_c)$ and $q_2(y,t_c)$ for position 1. The other curves show that the errors in the estimated results increase naturally when the positions of the sensors lines increase from the active surfaces of the specimen.

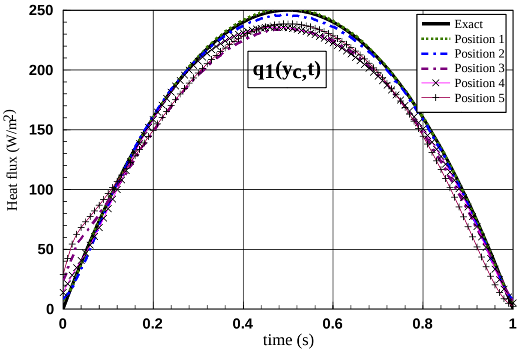

Figure 3a.
Evolution of exact and estimated heat flux $q_1(y,t)$ at $y = b/2$, for different positions.

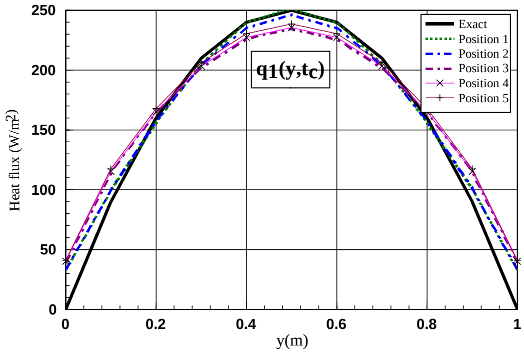

Figure 3c.
Exact and estimated profiles $q_1(y,t_c)$ at time $t_c = 0.5t_f$ for different positions.

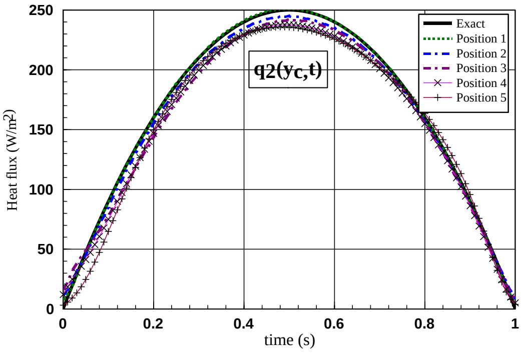

Figure 3b.
Evolution of exact and estimated heat flux $q_2(y_c,t)$ at $y_c = b/2$, for different positions.

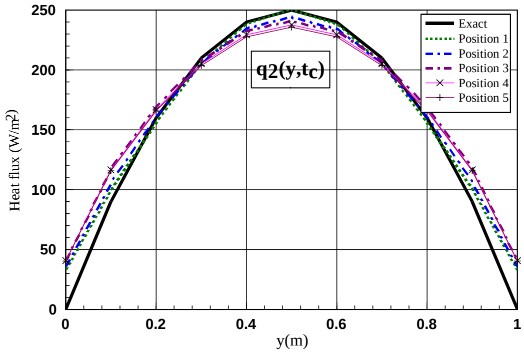

Figure 3d.
Exact and estimated profiles $q_2(y,t_c)$ at time $t_c = 0.5t_f$ for different positions.

For the same condition described above, we present in Figures 5a-b, the estimated results for nonsymmetrical positions of the sensor lines (L_α) and (L_β). The analyzed positions are presented in Table 2.

Table 2.

Position	(x_α, x_β)	ΔFo_1	ΔFo_2
1	$(0.1, 0.8)$	1	0.25
2	$(0.1, 0.6)$	1	0.625
3	$(0.1, 0.4)$	1	0.0278

The results show that the estimation of the heat flux $q_1(y,t)$ is better than the heat flux $q_2(y,t)$ and explain the importance of the position of the sensors towards the active surfaces. It should be noted however that the results obtained in this case for the first heat flux $q_1(y,t)$ are worse than those obtained with a symmetric configuration. This difference is explain by the fact that the estimations of the heat flux $q_1(y,t)$ are affected by those obtained for $q_2(y,t)$.

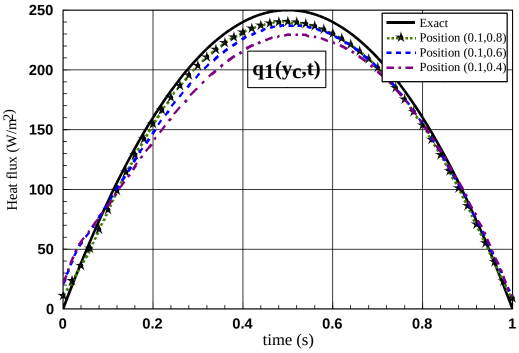

Figure 5a.
Evolution of exact and estimated heat flux $q_1(y_c,t)$ at $y_c = b/2$, for different nonsymmetrical positions.

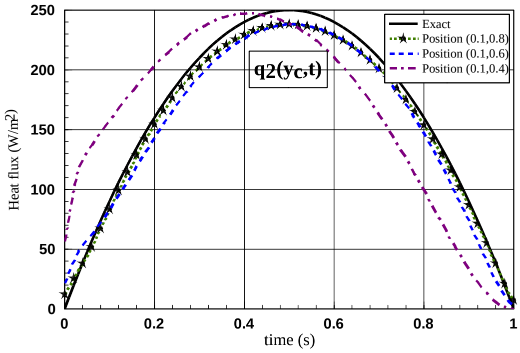

Figure 5b.
Evolution of exact and estimated heat flux $q_2(y_c,t)$ at $y_c = b/2$, for different nonsymmetrical positions.

4.2. INFLUENCE OF NOISY DATA

We present in figures 6a-b, for sensor lines positions $(0.1, 0.9)$, $N = 21$, $\Delta Fo_1 = \Delta Fo_2 = 1$, the estimated heat flux $q_1(y,t)$ for the four values of random noise $bx = 0.25$, 1, 3 and $5\,\%$ of the maximal of the simulated temperature. The corresponding residual criteria are presented in Table 3.

Table 3.

bx	0.25	1	3	5
δ^2	0.1063	1.7006	15.305	42.5138

The results show good agreement for small values of noise and the errors increase according to the noise. It is noted that the criteria of the iterative regularization method makes it possible to obtain acceptable and stable results for the amplitude of the noise imposed on the measured temperatures. The second estimated heat flux $q_2(y,t)$ shows the same behavior.

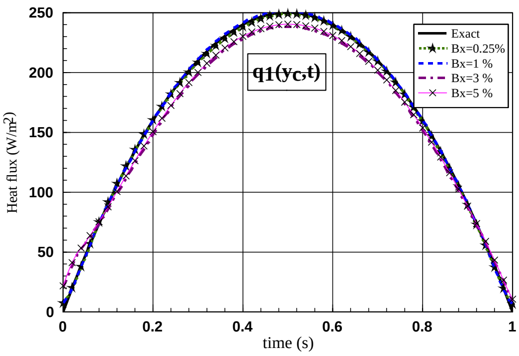

Figure 6a.
Influence of noisy data on the estimated heat flux $q_1(y_c,t)$.

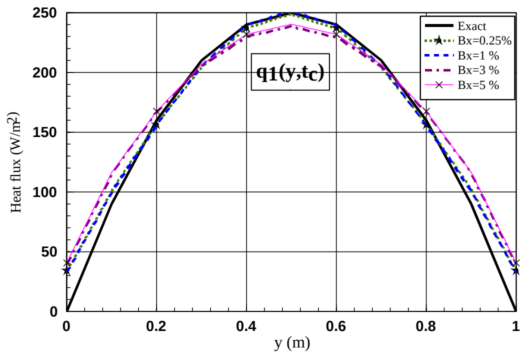

Figure 6b.
Influence of noisy data on the estimated heat flux $q_1(y,t_c)$.

4.3. NUMERICAL TESTS FOR CASE $q_1(y,t) \neq q_2(y,t)$

Figures 7a and b show a comparison between the exact and the estimated heat fluxes $q_1(y,t)$ and $q_2(y,t) = 0.25\,q_1(y,t)$ for two nonsymmetrical positions $(0.1,0.9)$ and $(0.3,0.9)$, with $bx = 1\%$, $N = 21$.

One finds the same remarks described above, i.e. as best estimate of the heat flux conditions nearest to the sensors. Here, the second heat flux presents a better estimate whereas that of first heat flux is less since the first line of the sensors is further away from the surface condition being estimated.

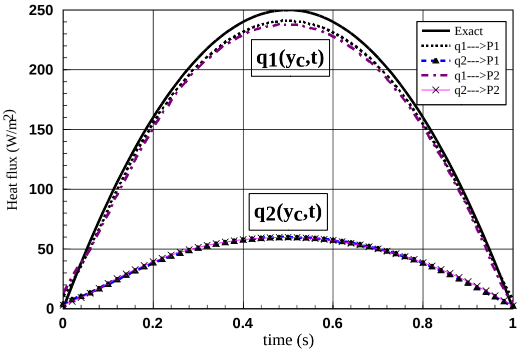

Figure 7a.
Evolution of estimated heat flux for case $q_1 \neq q_2$.

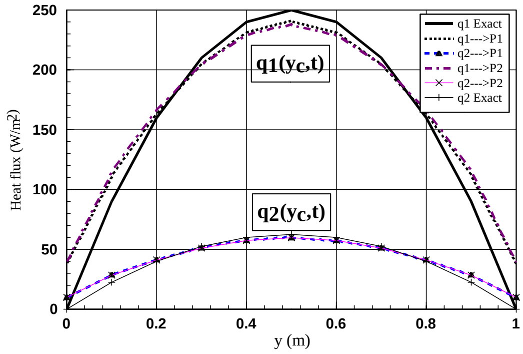

Figure 7b.
Distribution of estimated heat flux for case $q_1 \neq q_2$.

5. CONCLUSIONS

An inverse method is used for to estimate the unknown surface heat fluxes from temperature data measured at two lines parallel to the heated surfaces of a rectangular plate. The functional form of the unknown surface conditions is unknown a priori. The numerical algorithm of this inverse heat conduction problem is based on the iterative regularization method and on the conjugate gradient method. For each boundary heat flux, a descent parameter is computed and an optimal choice of the matrix of the descent parameters is used and shows an increase in the convergence rate.

The present estimates exhibit stable and accurate results and agree with the exact surface boundary conditions. Results show also that the accuracy of the estimated results decrease when the positions of the sensor line (or with delta Fourier numbers ($\Delta Fo \leq 0.05$)) increase from the unknown heated surface and also when the noise of measurement (or simulated) temperatures increase. Other results, not presented here, show a good agreement between exact and estimated results for different distribution forms (constant, sinusoidal, etc.) and in the case where the evolution of the distribution of the two unknown heat fluxes is different.

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